JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 3, May–June 1994

Nonlinear Predictive Controllers for Continuous Systems

Ping Lu
Iowa State University, Ames, Iowa 50011

In this paper a new technique for the design of nonlinear feedback controllers to track desired response histories is presented. The controller is developed based on continuous minimization of predicted tracking errors. Some properties on the tracking performance and robustness of the controller are established. Both state- and output-tracking problems are considered in a unified framework. The effectiveness of this approach is demonstrated by an application in missile autopilot design.

I. Introduction

IN many control problems of nonlinear dynamic systems, a reference trajectory $q(t)$ that represents the desired state of the system at each time is first obtained off line. This allows a thorough tradeoff study of various conflicting objectives and constraints, satisfaction of output requirements, and careful mission planning. Frequently $q(t)$ is a product of some type of optimization process; then the actual state of the dynamic system is controlled to follow the reference trajectory on line. Many complex aerospace systems, such as aircrafts, launch vehicles, and spacecraft, often operate in this way. The control problem amounts to finding a controller to drive the system to follow $q(t)$.

The state-tracking problem for robotic systems has seen impressive progress in recent years. Techniques based on optimal control[1,2] and variable-structure sliding control[3] have been proposed. However, all of these techniques are invariably based on some distinct features of the rigid-body dynamics of the robot. They include an independent control for each degree of freedom, a positive definite mass-inertia matrix, and some skew symmetric property. Many other nonlinear dynamic systems, notably aerospace vehicles, do not share this luxury. Thus, in the case of aerospace vehicle control, linear control techniques such as model following and linear quadratic controller design have been applied to the linearized model. Although the success has been tremendous, ever-increasing high-performance requirements now demand that a modern aircraft operate in regimes of large angles and angular rates where nonlinearities are dominant. The traditional linear control techniques are not adequate in those situations. The following simple example illustrates the inadequacy of the linearization technique and the lack of a nonlinear state-tracking technique.

Consider a subsonic aircraft flying in a horizontal plane. The speed of the aircraft is maintained at a constant $V = 150$ m/s by proper throttling. Assume a quadratic drag polar for the aircraft. The point-mass equations of motion are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = V\cos\psi \tag{1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = V\sin\psi \tag{2}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{g}{V}\tan\sigma \tag{3}$$

where x and y are position coordinates, ψ is the heading angle, and σ is the bank angle (treated as the control variable). The variable g is the gravitational acceleration at the flight altitude. Suppose that the reference state trajectory is specified by

$$x^*(t) = R\cos\omega t, \quad y^*(t) = R\sin\omega t$$

$$\psi^*(t) = \frac{\pi}{2} + \omega t, \quad 0 \le t \le 100 \text{ s} \tag{4}$$

where $R = 1000$ m and $\omega = V/R = 0.15$ rad/s. Clearly the reference trajectory is a circle of radius $R = 1000$ m centered at the origin. Given the initial state of the aircraft,

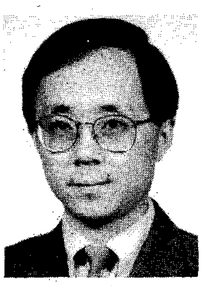
$$x(0) = y(0) = \psi(0) = 0 \tag{5}$$

we want to find the required control $\sigma(t)$ to bank the aircraft so that $x(t)$, $y(t)$, and $\psi(t)$ follow the desired history of Eq. (4). The variable σ is subject to the constraint

$$|\sigma| \le 80 \text{ deg} \tag{6}$$

The validity of linearization of the trajectory about the reference trajectory is questionable because of the large initial errors ($\Delta x_0 = 1000$ m and $\Delta\psi_0 = 90$ deg). Even when linearization is feasible, the linearized system will be time varying, which leaves little advantage for control law design. There is no readily available nonlinear state-tracking technique to apply.

An equally important problem closely related to state tracking is the output-tracking problem. In this case an output vector y that is a function of the state variables is defined for the system, and $q(t)$ represents the desired output. The output-tracking problem has been under intensive study. The



Dr. Ping Lu received his B.S. degree from the Beijing Institute of Aeronautics and Astronautics, China, in 1981 and his M.E. and Ph.D. degrees in aerospace engineering from the University of Michigan in 1984 and 1988, respectively. He worked as a research fellow from 1988 to 1989 at the University of Michigan. Since 1990, he has been an assistant professor in the Department of Aerospace Engineering and Engineering Mechanics at Iowa State University. His research interests include optimal control theory and its application and nonlinear control systems. He is a Senior Member of AIAA.

Received Aug. 15, 1992; revision received May 24, 1993; accepted for publication July 20, 1993.

prevalent approach is the feedback linearization approach using a differential geometric technique.[4] This technique involves nonlinear static feedback for canceling system nonlinearities and replaces the input/output dynamic behavior with desired linear dynamics. Feedback linearization applications in aerospace control are exemplified by the studies of Refs. 5–10. In this approach exact knowledge of the system nonlinearities is needed for cancellation; thus robustness is a practical concern. Also, when control saturation occurs, the feedback linearization control becomes irrelevant since feedback linearization is no longer achieved.

We present an alternative tracking controller design methodology in this paper. Section II describes the approach to obtain the controller on the basis of minimization of predicted tracking errors. Section III discusses tracking properties and the robustness of the controller. The output-tracking problem is treated in Sec. IV. In Sec. V a missile stabilization/control problem is solved using the theory developed. Section VI summarizes the work.

II. Controller Development

A. Problem Statement

We consider the system of the form

$$\dot{x}_1 = f_1(x) \tag{7}$$

$$\dot{x}_2 = f_2(x) + B_2(x)u \tag{8}$$

where $x_1 : R \rightarrow R^{n_1}$ and $x_2 : R \rightarrow R^{n_2}$, $n_1 + n_2 = n$, and $x(t) = [x_1^T(t)x_2^T(t)]^T \in X \subset R^n$ is the state vector. The $u(t) \in U \subset R^m$ represents the control. We assume that $m \leq n$ since this is the case for most physical systems. The X and U are some connected sets containing the origin; $f_1 : R^n \rightarrow R^{n_1}$ is sufficiently differentiable; and $f_2 : R^n \rightarrow R^{n_2}$ and $B_2 : R^n \rightarrow R^{n_2 \times m}$ only need to be C^1. Without loss of generality, we assume that none of the rows of $B_2(x)$ are constantly zero. For mechanical systems, Eq. (7) typically represents the kinematics in the system, as did Eqs. (1) and (2) in the example in Sec. I. Equation (8) is the dynamic part of the system; the kinematic part is usually well defined by physical relations. It is the dynamic part, Eq. (8), that is more complicated and more easily subject to uncertainties. The system can be rewritten in a conventional format

$$\dot{x} = f(x) + G(x)u \tag{9}$$

with

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \triangleq [g_1, g_2, \ldots, g_m]$$

We will assume that the reference trajectory, $q(t) = [q_1^T(t)q_2^T(t)]^T \in R^n$, $0 \leq t \leq t_f$, is generated from the same system model [Eqs. (7) and (8)] with some reference control $u^*(t)$

$$\dot{q}_1 = f_1(q) \tag{10}$$

$$\dot{q}_2 = f_2(q) + B_2(q)u^* \tag{11}$$

This assumption is needed for the discussion of the convergence properties in Sec. III for cases where $n > m$. In practical applications, it is not always necessary (Sec. V.C). The objective is to find a feedback control law $u = s(x, t)$ so that $x(t)$ tracks $q(t)$ for an arbitrary initial condition $x(0) \in X_0 \subset X$.

B. Continuous Predictive Controller

Let r_i, $i = 1, \ldots, n$, be the lowest order of derivative of x_i in which any component of u first appears at $x(t)$, $t \in [0, t_f]$. Define $v[x(t), h] = (v_1, \ldots, v_n)^T$, where

$$v_i = hL_f^0(f_i) + \frac{h^2}{2!}L_f^1(f_i) + \cdots + \frac{h^{r_i}}{r_i!}L_f^{r_i-1}(f_i), \quad i = 1, \ldots, n \tag{12}$$

where $h > 0$ is a real number, and f_i is the ith element of f. The $L_f^k(f_i)$ denotes the kth order Lie derivative of f_i with respect to f:

$$L_f^0(f_i) = f_i$$

$$L_f^1(f_i) = \frac{\partial f_i}{\partial x} f$$

$$L_f^2(f_i) = \frac{\partial L_f^1(f_i)}{\partial x} f$$

and so on. By state equation (9), at any instant $t \in [0, t_f]$, the current state $x(t)$ and control $u(t)$ determine the future state. If we expand each $x_i(t + h)$ for a small $h > 0$ in an r_ith order Taylor series at t, we can write $x(t + h)$ in a compact form,

$$x(t + h) \approx x(t) + v[x(t), h] + \Lambda(h)W[x(t)]u(t) \tag{13}$$

where $\Lambda(h) \in R^{n \times n}$ is a diagonal matrix with the elements on the main diagonal being

$$\lambda_{ii}(h) = \frac{h^{r_i}}{r_i!}, \qquad i = 1, \ldots, n \tag{14}$$

and $W(x) \in R^{n \times m}$ has each of its rows in the form

$$w_i\{L_{g_1}[L_f^{r_i-1}(x_i)], \ldots, L_{g_m}[L_f^{r_i-1}(x_i)]\}, \quad i = 1, \ldots, n \tag{15}$$

where the Lie derivative with respect to g_j is similarly defined

$$L_{g_j}[L_f^{r_i-1}(x_i)] = \frac{\partial L_f^{r_i-1}(x_i)}{\partial x} g_j, \qquad j = 1, \ldots, m$$

Obviously, by assumption for $i = n_1 + 1, \ldots, n$, $r_i = 1$, $v_i = hf_i$, and w_i = the ith row of G.

Consider a performance index that penalizes the tracking error at the next instant and current control expenditure

$$J[u(t)] = \tfrac{1}{2}[x(t + h) - q(t + h)]^T Q[x(t + h) - q(t + h)] + \tfrac{1}{2}u^T(t)Ru(t) \tag{16}$$

where $Q \in R^{n \times n}$ and $R \in R^{m \times m}$ are at least positive semidefinite. Similarly, we may expand the ith component of $q(t + h)$ in an r_ith order Taylor series,

$$q(t + h) \approx q(t) + d(t, h) \tag{17}$$

where the ith component of $d(t, h) \in R^n$ is

$$d_i(t, h) = h\dot{q}_i(t) + \frac{h^2}{2!}\ddot{q}_i(t) + \cdots + \frac{h^{r_i}}{r_i!}q_i^{(r_i)}(t), \quad i = 1, \ldots, n$$

Replace $x(t + h)$ and $q(t + h)$ in Eq. (16) by Eqs. (13) and (17), respectively. For given $x(t)$ and $q(t)$, the minimization of J with respect to $u(t)$ by setting $\partial J/\partial u(t) = 0$ yields an optimal predictive controller

$$u(t) = -\{[\Lambda(h)W(x)]^T Q\Lambda(h)W(x) + R\}^{-1} \times \{[\Lambda(h)W(x)]^T Q[e(t) + v(x, h) - d(t, h)]\} \tag{18}$$

where $e(t)x(t) - q(t)$ is the current tracking error.

When the dynamic system is linear and discrete in time, and h equals the sample period, performance index (16) reduces to the so-called one-step-ahead control formulation that was discussed by Goodwin and Sin.[11] In this respect, Eq. (18) may be viewed as a generalized nonlinear version of the one-step-ahead control for continuous systems. But the indefiniteness of the "step" h here can actually serve to provide some desired properties that are not shared by discrete systems, as we shall see shortly. Explicit formulas of Eq. (18) for two common cases are as follows.

Case 1. The $n_1 = 0$. In this case,

$$u(t) = -\frac{1}{h}[G^T(x)QG(x) + h^{-2}R]^{-1}G^T(x)Q\{e(t) + h[f(x) - \dot{q}(t)]\} \tag{19}$$

Case 2. The $n_1 \neq 0$, and $r_i = 2$ for $i = 1, \ldots, n_1$. Let

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & h^2 Q_2 \end{pmatrix}$$

where Q_1 and Q_2 are positive definite matrices of $n_1 \times n_1$ and $n_2 \times n_2$, respectively. Define

$$F_{11} = \frac{\partial f_1}{\partial x_1}, \qquad F_{12} = \frac{\partial f_1}{\partial x_2}$$

Equation (18) becomes

$$u(t) = -P\left(\frac{1}{2h^2}(F_{12}B_2)^T Q_1\left\{e_1 + h\dot{e}_1 + \frac{h^2}{2}[F_{11}f_1 + F_{12}f_2 - \ddot{q}_1(t)]\right\} + \frac{1}{h}B_2^T Q_2\{e_2 + h[f_2 - \dot{q}_2(t)]\}\right) \tag{20}$$

where

$$P = [\tfrac{1}{4}(F_{12}B_2)^T Q_1 F_{12}B_2 + B_2^T Q_2 B_2 + h^{-4}R]^{-1}$$
$$e_i(t) = x_i(t) - q_i(t), \qquad i = 1, 2 \tag{21}$$

III. Controller Evaluation

We shall discuss in this section some important properties of the controller derived earlier and some possible modifications. At the risk of confusing a function with the function value, the argument of a function is frequently dropped throughout the rest of the paper for simplicity of notation.

A. Tracking Properties

The foremost concerned property will be the tracking capability. To begin with, we state a desired property of the controller: If there is no initial tracking error [$e(0) = 0$], controller (18) will maintain a perfect tracking for all $t \in [0, t_f]$, provided that $R = 0$ and $W[q(t)]$ is of full rank. This can be seen by verifying with the aid of Eqs. (10) and (11) that the kth order derivative of $q_i(t)$ for $k < r_i$ is

$$q_i^{(k)}(t) = \frac{\partial^{k-1} f_i[q(t)]}{\partial q^{k-1}} f[q(t)] = L_f^{k-1}(f_i)[q(t)]$$
$$k = 1, \ldots, r_i - 1, \qquad i = 1, \ldots, n$$

and for $k = r_i$,

$$q_i^{(r_i)}(t) = L_f^{r_i-1}(f_i)[q(t)] + w_i^T[q(t)]u^*(t), \qquad i = 1, \ldots, n$$

Thus, when $x(0) = q(0)$ [$e(0) = 0$], $v - d = -\Lambda(h)W[q(0)]u^*(0)$ by the definitions of v and d. Accordingly, Eq. (18) gives

$$u(0) = (\{\Lambda(h)W[q(0)]\}^T Q\Lambda(h)W[q(0)] + R)^{-1} \times (\{\Lambda(h)W[q(0)]\}^T Q\{\Lambda(h)W[q(0)]\})u^*(0) \tag{22}$$

If the $n \times m$ matrix $W[x(0)]$ has full rank, so does the $m \times m$ matrix $(\Lambda W)^T Q\Lambda W$ (since $m \leq n$). Letting $R = 0$ in Eq. (22) immediately gives $u(0) = u^*(0)$. Thus $x(\Delta t) = q(\Delta t)$, and $u(\Delta t) = u^*(\Delta t)$ for an infinitesimal $\Delta t > 0$. Extending this process gives $x(t) = q(t)$ for all $t \in [0, t_f]$. Note that this property is valid for a general system for which n_1, n_2, and m can be arbitrary.

When $e(0) \neq 0$, asymptotic convergence of $e(t)$ to zero is desired. To examine if the convergence is possible, let us first consider the following class of systems. Assume that $n_1 = 0$, $n_2 = n = m$, and G has full rank at $x(t) \in X$. This is case 1 in the preceding section. Controller (19) applies. Let $R = 0$. Then for any $Q > 0$ and $q(t)$, we have

$$\dot{e} = -\frac{1}{h}e(t) \tag{23}$$

That is, globally asymptotic tracking of $q(t)$ is achieved with h being the time constant of the error dynamics.

Next, let us consider another class of systems for which the dynamic equations are

$$\dot{x}_1 = Cx_2 \tag{24}$$

$$\dot{x}_2 = f_2(x) + B_2(x)u \tag{25}$$

In this case we assume that $n_1 = n_2 = n/2 = m$, C is a nonsingular constant matrix, and $B_2(x)$ is of full rank for all x. System (24) and (25) belongs to case 2. The dynamics of a rigid-body spacecraft with momentum gyroscopes as the controls, for instance, are in this class.[14] Rigid-body robots also fall into this class of systems. We apply controller (20) with $R = 0$, which in this case is

$$u(t) = -B_2^{-1}\bar{P}B_2^{-T}\left(\frac{1}{2h^2}(CB_2)^T Q_1\left\{e_1 + h\dot{e}_1 + \frac{h^2}{2}[Cf_2 - C\dot{q}_2(t)]\right\} + \frac{1}{h}B_2^T Q_2\{e_2 + h[f_2 - \dot{q}_2(t)]\}\right) \tag{26}$$

where, according to Eqs. (10) and (24), $\ddot{q}_1(t) = C\dot{q}_2(t)$ has been used in Eq. (26), and

$$\bar{P} = (\tfrac{1}{4}C^T Q_1 C + Q_2)^{-1} > 0 \tag{27}$$

Substituting Eq. (26) into Eq. (25) yields the error dynamics

$$\dot{e}_1 = Ce_2 \tag{28}$$

$$\dot{e}_2 = -\frac{1}{2h^2}\bar{P}C^T Q_1 e_1 - \frac{1}{h}\bar{P}(\tfrac{1}{2}C^T Q_1 C + Q_2)e_2 \tag{29}$$

To study the stability of Eqs. (28) and (29), we consider a Lyapunov function candidate

$$V = \frac{1}{4h}e_1^T Q_1 e_1 + \frac{h}{2}e_2^T \bar{P}^{-1} e_2 > 0, \qquad e \neq 0 \tag{30}$$

Thus, for all e,

$$\dot{V} = -e_2^T(\tfrac{1}{2}C^T Q_1 C + Q_2)e_2 \leq 0 \tag{31}$$

By the LaSalle Theorem,[12,13] the solution $e(t)$ of Eqs. (28) and (29) tends to the invariant set

$$S = (e \mid e_2 = 0, \quad \bar{P}C^T Q_1 e_1 = 0) \tag{32}$$

Since the $n_1 \times n_1$ matrix $\bar{P}C^T Q_1$ is of full rank, $\bar{P}C^T Q_1 e_1 = 0 \Rightarrow e_1 = 0$. Therefore, $S = \{0\}$. So $e = 0$ is globally asymptotically stable.

A special case of tracking is the problem of stabilization in which the reference trajectory $q(t)$ is a constant vector. The previous results thus also establish the closed-loop stability of

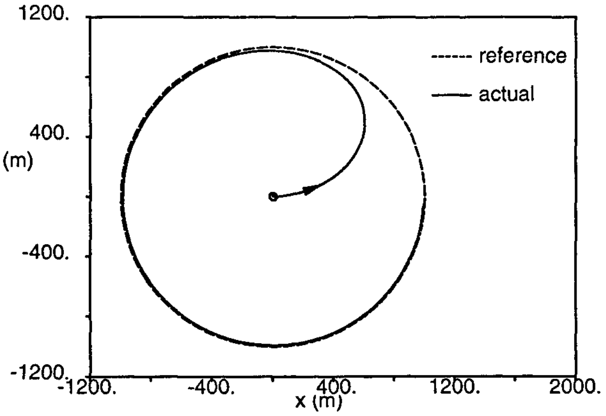


Fig. 1 Aircraft tracking trajectory in the x-y plane.

nonlinear feedback stabilization of those systems.

To demonstrate how this technique is applied to a more general system, we return to the example in Sec. I. Let the control $u = \tan \sigma$. The system of Eqs. (1–3) fits to use controller (20). Note that the degree of freedom is two, and there is only one control. Use Eq. (4) as the reference trajectory and apply control law (20), which gives

$$u = \tan \sigma = \frac{-1}{P} \left\{ \frac{1}{2h^2} [-Q_1(\Delta x + h\Delta \dot{x})\sin \psi + Q_2(\Delta y + h\Delta \dot{y})\cos \psi] - 0.25R\omega^2(Q_1 \sin \psi \cos \omega t - Q_2 \cos \psi \sin \omega t) + \frac{Q_3}{hV}(\Delta\psi - h\omega) \right\} \tag{33}$$

where $Q_i > 0$, $i = 1, 2, 3$, are the weighting coefficients, $R = 1000$ m, $\omega = 0.15$ rad/s, and

$$P = 0.25g(Q_1 \sin^2\psi + Q_2 \cos^2\psi) + \frac{gQ_3}{V^2} \tag{34}$$

Figure 1 shows the tracking trajectory in the x-y plane with a dashed line for $Q_i = 1$ and $h = 0.5$. Despite its large initial errors, the trajectory merges onto the reference trajectory (the circle in dashed line) very quickly and smoothly. Note that this is a dynamic tracking problem, since $x^*(t)$, $y^*(t)$, and $\psi^*(t)$ are all functions of time.

B. Robustness

When system uncertainties are present, it is desired that the controller maintains satisfactory performance. Controller (18) is derived based on continuous minimization of the errors between the controlled variables and their desired values. This mechanism provides a certain degree of robustness in some cases. More generally, the structure of the controller is such that it allows the controller to achieve robustness through high-gain feedback. To illustrate this prospective, we consider unmodeled dynamics $\Delta f(x)$ and $\Delta G(x)$ in the system (9):

$$\dot{x} = f(x) + \Delta f(x) + [G(x) + \Delta G(x)]u \tag{35}$$

Let us assume that $m = n$, that $G[x(t)]$ is of full rank at $x(t)$ (case 1 in Sec. II.B), and that the unmodeled dynamics in Eq. (35) satisfy the following conditions, for all $x \in X$

$$\|\Delta f(x)\| < N_1 \tag{36}$$

$$\Delta G(x) = \delta(x)G(x), \qquad -1 < \delta(x) < N_2 \tag{37}$$

where N_1 and N_2 are positive constants. Moreover, we assume that $\|f(x)\|$ and $\|\dot{q}(t)\|$ are bounded for all $x \in X$ and $t \in [0, t_f]$. The controller is said to be robust in the presence of the modeling uncertainties (36) and (37) if, for any given $\epsilon > 0$, the tracking error can be made to satisfy $\|e(t)\| < \epsilon$ for sufficiently small initial error. We apply control law (19), which is based on the nominal model (9). Using Eq. (19) with $R = 0$ in Eq. (35) produces the error dynamics

$$\dot{e}(t) = -\frac{1}{h}[1 + \delta(x)]e(t) + \Delta f(x) - \delta(x)[f(x) - \dot{q}] \tag{38}$$

Change the independent variable by $\tau = t/h$. Then

$$e' = \frac{de}{d\tau} = -[1 + \delta(x)]e + h\{\Delta f(x) - \delta(x)[f(x) - \dot{q}]\} \tag{39}$$

Since $f(x)$ and $\dot{q}(t)$ are bounded for all $x \in X$ and $t \in [0, t_f]$, by the assumptions of Eqs. (36) and (37) the quantity inside the brace is bounded. For any given $\epsilon > 0$ and $\eta_1(\epsilon) > 0$, one can always find an $h_{\max}$ such that, for all $0 < h < h_{\max}$, $\|h\{\Delta f(x) - \delta(x)[f(x) - \dot{q}]\}\| < \eta_1$. Since the condition $-1 < \delta(x)$ guarantees asymptotic stability of the system $e' = -[1 + \delta(x)]e$, by Malkin's theorem,[12,15] there exists an $\eta_2(\epsilon) > 0$ such that, for $\|e(0)\| < \eta_2$, $\|e(t)\| < \epsilon$ for all $t \geq 0$. Note that small h means high controller gain from Eq. (19). In fact, explicit formulas that determine what values of h should be used for a given tracking accuracy ϵ can be derived and will be reported in the near future.

For system (24) and (25), if there are unmodeled dynamics $\Delta f_2(x)$ and $\Delta B_2(x)$ in Eq. (25) that satisfy Eqs. (36) and (37), it can be similarly shown that controller (20) can also maintain tracking accuracy $\|e(t)\| < \epsilon$ for any $\epsilon > 0$.

IV. Output Tracking

The approach developed in the preceding sections can be readily extended to the output-tracking problem. Suppose that, in addition to the system state equations (7) and (8) or (9), we have output equations

$$y = c(x) \tag{40}$$

where $y \in R^l$ and $c: R^n \to R^l$ is sufficiently differentiable. Let $q(t)$, $0 \leq t \leq t_f$, denote the reference output in this section.

A. Output-Tracking Controller

Let r_i, $i = 1, \ldots, l$, be the lowest order of derivative of y_i in which any component of u first appears at $x(t)$, $t \in [0, t_f]$. In exactly the same way as in Sec. II.B, we expand each $y_i(t + h)$ and $q_i(t + h)$ in a Taylor expansion of r_ith order. Then by minimizing a performance index

$$J[u(t)] = \tfrac{1}{2}[y(t + h) - q(t + h)]^T Q[y(t + h) - q(t + h)] + \tfrac{1}{2}u^T(t)Ru(t) \tag{41}$$

we obtain an output-tracking control law

$$u(t) = -\{[\Lambda(h)W(x)]^T Q\Lambda(h)W(x) + R\}^{-1} \times \{[\Lambda(h)W(x)]^T Q[e(t) + z(x, h) - d(t, h)]\} \tag{42}$$

where $Q \in R^{l \times l}$ is positive definite and $R \in R^{m \times m}$ at least positive semidefinite; $e(t) = y(t) - q(t)$; $\Lambda(h) \in R^{l \times l}$ is a diagonal matrix with the elements on the main diagonal being

$$\lambda_{ii}(h) = \frac{h^{r_i}}{r_i!}, \qquad i = 1, \ldots, l \tag{43}$$

the ith component of $z \in R^l$ is

$$z_i(x, h) = hL_f(c_i) + \frac{h^2}{2!}L_f^2(c_i) + \cdots + \frac{h^{r_i}}{r_i!}L_f^{r_i}(c_i), \qquad i = 1, \ldots, l \tag{44}$$

and the ith component of d is defined as before

$$d_i(t, h) = h\dot{q}_i(t) + \frac{h^2}{2!}\ddot{q}_i(t) + \cdots + \frac{h^{r_i}}{r_i!}q_i^{(r_i)}(t) \qquad i = 1, \ldots, l \tag{45}$$

The ith row of the $l \times m$ matrix W takes the form

$$w_i = \{L_{g1}[L_f^{r_i-1}(c_i)], \ldots, Lg_m[L_f^{r_i-1}(c_i)]\}, i = 1, \ldots, l \tag{46}$$

When some state variables are the outputs, say, without loss of generality, $y_i = x_i$, $i = 1, \ldots, l$, $l \leq n$, the state-tracking controller Eq. (18) and the output-tracking controller Eq. (42) are the same if one sets $Q_{jj} = 0$ in Eq. (18), $j = l + 1, \ldots, n$, where the various Q_{jj} are the elements on the main diagonal of the Q matrix. This feature enables a controller in some cases to function as both state- and output-tracking controllers by switching on and off the parameters Q_{jj} (Secs. V.B and V.C). When to use state tracking and when to use output tracking are largely determined by whether or not all state variables need to be controlled to achieve satisfactory performance.

B. Tracking Performance

In this section we shall make the standard assumption that the number of outputs is equal to that of the controls, i.e., $l = m$. Moreover, we assume that $W(x)$ is nonsingular for all $x \in X$. Let $R = 0$ in the controller (42). Differentiating y_i r_i-times and using Eq. (42), we can show that the ith tracking error dynamics is

$$\frac{h^{r_i}}{r_i!}e_i^{(r_i)} + \frac{h^{r_i-1}}{(r_i-1)!}e_i^{(r_i-1)} + \cdots + h\dot{e}_i + e_i = 0 \tag{47}$$

Equation (47) is linear, time invariant. We see that the proposed tracking controller design technique naturally leads to feedback linearization. For most of the mechanical systems with actuator outputs as the controls, $r_i = 1$ or 2. In these cases, the eigenvalues of the error dynamics are

$$s_1 = -\frac{1}{h}, \qquad \text{if } r_i = 1 \tag{48}$$

$$s_{1,2} = \frac{1}{h}(-1 \pm j), \qquad \text{if } r_i = 2 \tag{49}$$

Note that the damping ratio of the complex roots in Eq. (49) is 0.707, a well-accepted best choice. Note that, as well known in the output-tracking problem, the system needs to be internally stable (minimum phase) to prevent the uncontrolled state variables to become unbounded.[4]

We should point out an important advantage of the current control law (42) over the input/output linearization control when control saturation is encountered. When the feedback linearization control saturates, it is not clear at all that one should keep using the saturated control since the input/output linearization is not achievable any way. In contrast, when the control computed from Eq. (42) exceeds the control bounds, the use of the maximum (or minimum) value of the control still has a clear physical meaning: it is still the best choice within the control bounds that minimizes the performance index J in Eq. (41).

By Sec. IV.A, we can write m output dynamic equations for $y_i^{(r_i)}$, $i = 1, \ldots, m$, as

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} L_f^{r_1}(c_1) \\ \vdots \\ L_f^{(r_m)}(c_m) \end{bmatrix} + W(x)u \triangleq p(x) + W(x)u \tag{50}$$

If there are unmodeled dynamics $\Delta p(x)$ and $\Delta W(x)$ similar to conditions (36) and (37), by similar analysis we can conclude that the robustness discussion in Sec. III.B is applicable to the output-tracking problem.

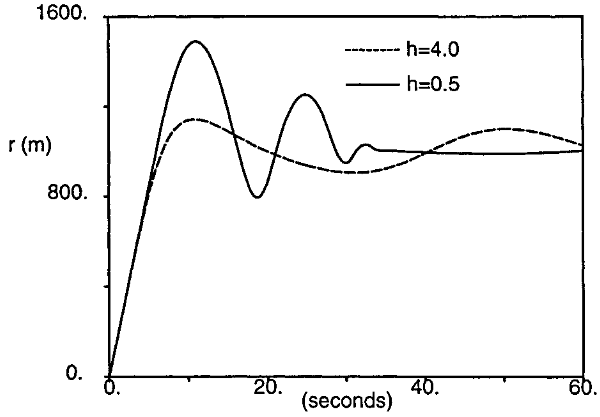


Fig. 2 Flight radii of the aircraft in the presence of wind.

To demonstrate the robustness of the tracking controller with high-gain feedback, we again use the example in Sec. I. Suppose that there is an unmodeled wind with constant velocity. The two equations of motion affected are

$$\frac{dx}{dt} = V\cos\psi + w_x \tag{51}$$

$$\frac{dy}{dt} = V\sin\psi + w_y \tag{52}$$

where w_x and w_y are the constant wind velocity components in the x and y directions, respectively. Suppose that the objective is to fly the aircraft in a circle with radius $R = 1000$ m centered at the origin. Choose the output as

$$r = x^2 + y^2 \tag{53}$$

The reference output then is $q = R^2 = 10^6$ m^2. For this output the index $r_i = 2$. It is straightforward to show that the resulting unmodeled dynamics in the $\ddot{r}$ equation are

$$\Delta p = 2[\cos\psi(1 + V)w_x + \sin\psi(1 + V)w_y + w_x^2 + w_y^2] \tag{54}$$

Clearly $|\Delta p|$ is bounded by a constant for all x, y, and ψ. The output-tracking controller based on Eq. (42) for the nominal dynamics (no wind) is

$$u = \tan\sigma = -\frac{1}{gh^2}\frac{(e + h\dot{e} + h^2V^2)}{(y\cos\psi - x\sin\psi)} \tag{55}$$

For $w_x = w_y = 15$ m/s, Fig. 2 shows two histories of the flight radius $\sqrt{x^2 + y^2}$ with two different values of h as used in Eq. (55). The dashed line shows that when $h = 4.0$ the maximum steady-state tracking error $|e|_{max} = \max|\sqrt{x^2+y^2} - 1000|$ is about 100 m. When the controller gain is increased by 16 times by reducing h to 0.5 [since $1/h^2$ is the controller gain by Eq. (55)], the solid line tells that the steady state $|e|_{max}$ is less than 10 m. Although it is not shown in the figure, in both cases the control $\tan\sigma$ is initially saturated to its maximum value.

V. Missile Autopilot Design

A. Missile Model

Extensive applications of the previous design techniques to the flight control of a high-performance fighter aircraft will be reported in another paper. In this section we shall be content to demonstrate the practical usefulness of the approach with the following missile control problem.

Consider the longitudinal rigid-body dynamics of a missile. The model is taken from Ref. 16 for a missile traveling at Mach 3 at an altitude of 6096 m (20,000 ft):

$$\dot{\alpha} = \frac{180gQS}{\pi WV}\cos(\pi\alpha/180)\phi_n(\alpha) + q + \frac{180gQS}{\pi WV}\cos(\pi\alpha/180)b_n\delta \tag{56}$$

$$\dot{q} = \frac{180QSd}{I_{yy}}\phi_m(\alpha) + \frac{180QSd}{I_{yy}}b_m\delta \tag{57}$$

where α is the angle of attack (deg), q the pitch rate (deg/s), δ the fin deflection (deg), g the gravitational acceleration (9.8 m/s), W the weight (4410 kg), V the speed (947.6 m/s), I_{yy} the pitch moment of inertia (247.44 kg-m^2), Q the dynamic pressure (293,638 N/m^2), S the reference area (0.04087 m^2), and d the reference diameter (0.229 m). The normal force and pitch moment aerodynamic coefficients for $|\alpha| \leq 20$ deg are given by

$$C_n = \phi_n(\alpha) + b_n\delta = 0.000103\alpha^3 - 0.00945\alpha|\alpha| - 0.17\alpha - 0.034\delta \tag{58}$$

$$C_m = \phi_m(\alpha) + b_m\delta = 0.000215\alpha^3 - 0.0195\alpha|\alpha| + 0.051\alpha - 0.206\delta \tag{59}$$

For convenience, we rewrite Eqs. (56) and (57) as

$$\dot{\alpha} = f_1(\alpha) + q + b_1(\alpha)\delta \tag{60}$$

$$\dot{q} = f_2(\alpha) + b_2\delta \tag{61}$$

with $f_1(\alpha)$, $f_2(\alpha)$, $b_1(\alpha)$, and constant b_2 obviously defined. The tail fin actuator dynamics are approximated by a first-order lag

$$\dot{\delta} = \frac{1}{\tau}(-\delta + u) \tag{62}$$

where u represents the commanded fin deflection (deg), and $\tau > 0$ the time constant. The fin deflection is limited by

$$|\delta| \leq 30 \text{ deg}$$

The system equations are now Eqs. (60–62). It can be readily verified by examining the linearized equations of system (60–62) that the system is open loop unstable at the equilibrium point $\alpha = q = \delta = 0$. The first control objective is to design an autopilot that stabilizes the missile.

B. Nonlinear Feedback Stabilization

Following the notation in Sec. II.B, we denote $x_1 = (\alpha\ q)^T$ and $x_2 = \delta$. The reference trajectory is $q(t) = 0$. The system falls into the category of case 2 in Sec. II.B. The physical variables to be controlled are α and q. So, following state-tracking controller Eq. (20) and setting $Q_2 = 0$, $Q_1 = \text{diag}(Q_1, Q_2)$, and $R = 0$, we arrive at a nonlinear feedback control law

$$u = \delta - \frac{\tau}{Q_1b_1^2 + Q_2b_2^2}\left\{\frac{2}{h^2}(Q_1b_1\alpha + Q_2b_2q) + \left[\frac{2}{h}Q_1b_1 + (f_{1\alpha} + b_{1\alpha}\delta)Q_1b_1 + Q_2b_2f_{2\alpha}\right](f_1 + q + b_1\delta) + \left(\frac{2}{h}Q_2b_2 + Q_1b_1\right)(f_2 + b_2\delta)\right\} \tag{63}$$

where

$$f_{1\alpha} = \frac{\partial f_1(\alpha)}{\partial\alpha}, \qquad f_{2\alpha} = \frac{\partial f_2(\alpha)}{\partial\alpha}, \qquad b_{1\alpha} = \frac{\partial b_1(\alpha)}{\partial\alpha}$$

For computation, $Q_1 = Q_2 = 1$, $\tau = 0.1$, and $h = 0.05$ are chosen. It can be shown by using the Lyapunov indirect method[13] that under control law (63) the missile closed-loop dynamics are asymptotically stable at (0, 0, 0). To test the effectiveness of the design, we choose the initial conditions

$$\alpha(0) = 10 \text{ deg}, \qquad q(0) = 20 \text{ (deg/s)}, \qquad \delta(0) = 0 \text{ deg} \tag{64}$$

The variations of α and q are shown in Figs. 3 and 4 in solid lines. Since appreciable modeling errors in the missile aerodynamic coefficients are expected to be present, the controller is also tested for cases in which the aerodynamic coefficients C_n and C_m simultaneously have +25 and then −25% perturbations. The corresponding α and q histories are also plotted in Figs. 3 and 4. It is seen that the controller performs remarkably well in all cases.

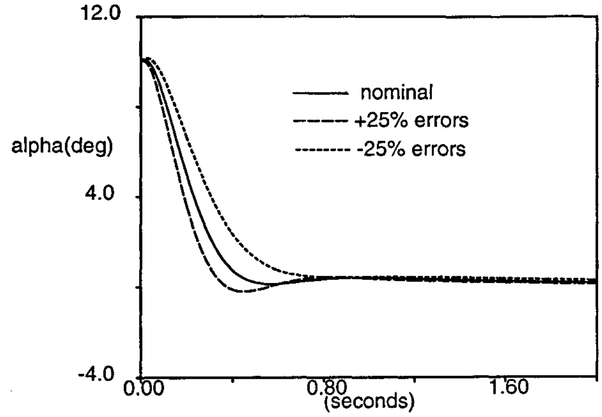


Fig. 3 Angle of attack histories of the missile corresponding to nominal and perturbed aerodynamic coefficients.

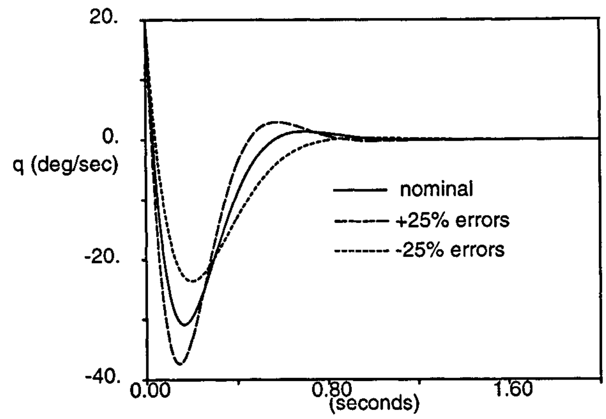


Fig. 4 Pitch rate histories of the missile corresponding to nominal and perturbed aerodynamic coefficients.

C. Command Tracking

When the missile is desired to track a given α or q command, the problem becomes an output-tracking problem. The same controller (63) can readily become the required tracking controller with some minor adjustments. To see this, let us assume that α-command tracking is desired. Let α_{com} be the command input (desired output). Following the procedure in Sec. IV.A gives a controller that is the same by setting $Q_2 = 0$ and replacing α by $\alpha - \alpha_{com}$ in Eq. (63):

$$u = \delta - \frac{\tau}{b_1}\left[\frac{2}{h^2}(\alpha - \alpha_{com}) + \left(\frac{2}{h} + f_{1\alpha} + b_{1\alpha}\delta\right) \times (f_1 + q + b_1\delta) + f_2 + b_2\delta\right] \tag{65}$$

Figure 5 shows the α response to a square wave α_{com} varying between 16 and 8 deg. The corresponding history of the fin

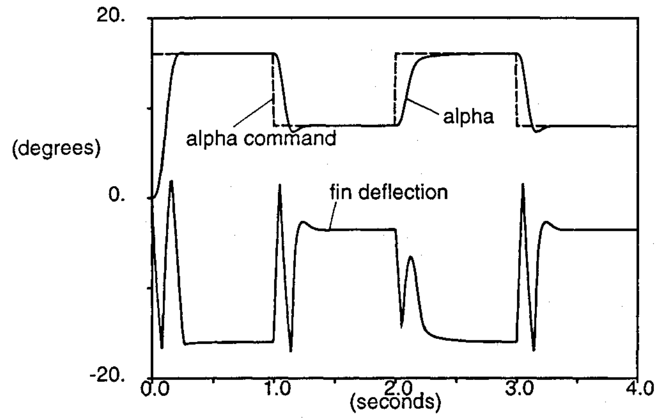


Fig. 5 Angle of attack and tail fin deflection responses of the missile to a square-wave command.

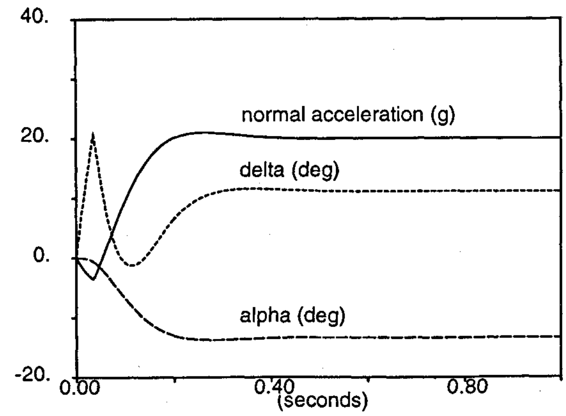


Fig. 6 Step responses to a normal acceleration command of 20 *g*.

deflection is also depicted in the same figure. Again, the controller proves quite satisfactory. Note that in this case α_{com} does not exactly satisfy the dynamic equation (56).

Controller (65) can be used to control the normal acceleration of the missile by commanding α. The normal acceleration is given in g by

$$a_n = \frac{QSC_n}{W} \tag{66}$$

During the pursuit of a maneuverable target, the missile autopilot controls the angle of attack to generate a normal acceleration according to the command from the guidance law. Reference 17 discusses an approach using linear parameter variation for gain scheduling and μ synthesis for design of an inner-outer loop configuration for the autopilot. Our nonlinear controller (65) offers a particularly simple solution to this challenging problem. For a specified value of the normal acceleration, denoted by $a_{n_{com}}$, at steady state $\dot{q} = 0$, which from Eq. (61) results in

$$\delta = -f_2(\alpha)/b_2 \tag{67}$$

Using Eq. (67) in the expression for C_n [Eq. (58)] and replacing C_n in Eq. (66), we have

$$\phi_n(\alpha) - \frac{b_n}{b_m}\phi_m(\alpha) - \frac{a_{n_{com}}W}{QS} = 0 \tag{68}$$

One of the roots of Eq. (68) is the corresponding required angle of attack for this specified $a_{n_{com}}$. This α value then can be used as the α_{com} in controller (65). In the steady state, the normal acceleration will have the commanded value. Figure 6 displays the step response of the normal acceleration to $a_{n_{com}} = 20g$. The corresponding $\alpha_{com} = -3.36$ deg. It is seen that the rise time is about 0.2 s, and the overshoot and steady-state error are negligible. The responses of α and fin deflection δ are also shown in the same figure. In implementation, a number of the appropriate roots of Eq. (68) for various values of $a_{n_{com}}$ can be computed off line and stored. For a time-varying normal acceleration command $a_{n_{com}}(t)$, a time-varying angle of attack command $\alpha_{com}(t)$ may be generated by interpolation over $a_{n_{com}}(t)$.

Finally, it should be noted that, although using the normal acceleration a_n as an output and then employing the output controller discussed in Sec. IV seems most straightforward, it is not a viable way to control a_n. This is because, with a_n as the output, the internal dynamics of the missile are unstable. The internal dynamics of the system, or referred to as zero dynamics alternatively,[4] are obtained from the system equations by setting $a_n \equiv 0$, which leads to

$$\ddot{\alpha} = 635.4(-0.00048\alpha^3 + 0.0445\alpha|\alpha| + 1.2\alpha) \tag{69}$$

Obviously, this system is not stable at the origin. So α and q can become unbounded even if a_n is under control. In contrast, following similar steps we can easily show that the zero dynamics of the missile are stable when α is the controlled output.

VI. Conclusions

A new design methodology for nonlinear feedback controllers that yield satisfactory tracking performance for a class of dynamic systems is proposed. The controller is developed based on continuous minimization of predicted tracking errors. With the proposed formulation, the state- and output-tracking problems can be treated in the same framework. This flexibility allows the same controller to be used for both state- and output-tracking purposes by turning on and off some controller parameters. For the output-tracking problem, the approach naturally leads to input/output linearization. Some asymptotic tracking convergence properties have been established. It is shown that the controller can maintain robust performance in the presence of unmodeled dynamics via high-gain feedback. As an application of this technique, the problem of design of a nonlinear missile autopilot is solved. First, the open-loop unstable missile is stabilized with a robust feedback controller using the technique developed. Then, with minor adjustments, the same autopilot is shown to be very effective in controlling the motion of the missile. In particular, the normal acceleration commands from the guidance law can be executed accurately.

References

[1]Luo, G. L., and Saridis, G. N., "L-Q Design of PID Controller of Robot Arms," *IEEE Journal of Robotics and Automation,* Vol. RA-1, No. 3, 1985, pp. 152–159.

[2]Johansson, R., "Quadratic Optimization of Motion Coordination and Control," *IEEE Transactions on Automatic Control,* Vol. 35, No. 11, 1990, pp. 1197–1210.

[3]Slotine, J. J. E., and Sastry, S. S., "Tracking Control of Nonlinear Systems Using Sliding Surfaces with Applications to Robot Manipulators," *International Journal of Control,* Vol. 38, No. 2, 1983, pp. 465–492.

[4]Isidori, A., *Nonlinear Control Systems: An Introduction,* 2nd ed., Springer-Verlag, New York, 1989.

[5]Meyer, G. R., Su, R., and Hunt, R. L., "Application of Nonlinear Transformations to Automatic Flight Control," *Automatica,* Vol. 20, 1984, No. 1, pp. 103–107.

[6]Menon, P. K. A., Badgett, M. E., Walker, R. A., and Duke, E. L., "Nonlinear Flight Test Trajectory Control for Aircraft," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 1, 1987, pp. 67–72.

[7]Lane, S. H., and Stengel, R. F., "Flight Control Design Using Nonlinear Inverse Dynamics," *Automatica,* Vol. 24, No. 4, 1988, pp. 471–483.

[8]Snell, S. A., Enns, D. F., and Garrard, W. L., "Nonlinear Inversion Flight Control for a Supermaneuverable Aircraft," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 4, 1992, pp. 976–984.

[9]Van Buren, M. A., and Mease, K. D., "Aerospace Plane Guidance

Using Time-Scale Decomposition and Feedback Linearization," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 5, 1992, pp. 1166–1174.

[10]Wise, K. A., "Nonlinear Aircraft Control Using Dynamic Inversion," *Proceedings of American Control Conference,* Vol. 2, 1992, pp. 1322–1326.

[11]Goodwin, G. C., and Sin, K. S., *Adaptive Filtering, Prediction and Control,* Prentice-Hall, Englewood Cliffs, NJ, 1984.

[12]La Salle, J., and Solomon, L., *Stability by Lyapunov's Direct Method with Application,* Academic Press, New York, 1961.

[13]Vidyasagar, M., *Nonlinear Systems Analysis,* Prentice-Hall, Englewood Cliffs, NJ, 1978.

[14]Parlos, A. G., and Sunkel, J. W., "Adaptive Attitude Control and Momentum Management for Large-Angle Spacecraft Maneuvers," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 4, 1992, pp. 1018–1028.

[15]Malkin, I. G., "Questions Concerning Transformation of Lyapunov Theorem on Asymptotic Stability," *Prikladnaia Matematika i Mekhanika,* Vol. 18, 1954, pp. 129–138.

[16]Reichert, R., "Modern Robust Control for Missile Autopilot Design," *Proceedings of American Control Conference* (San Diego, CA), June 1990.

[17]Shamma, J. S., and Cloutier, J. R., "Gain-Scheduled Missile Autopilot Design Using Linear Parameter Varying Transformations," *Journal of Guidance, Control, and Dynamics,* Vol. 16, No. 2, 1993, pp. 256–263.